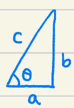
Chap 1. The genesis of Fourier Analysis.

(起源)

- The idea of Fourier analysis (series) was first used by Bernoulli (1753) to study wave equation and by Fourier (1807) to study heat equation. Below we will derive wave equation and illustrate how one can apply the method of Fourier analysis.

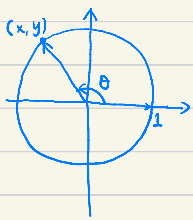
1.1 Sine and cosine functions.



$\sin\theta = \frac{b}{c}$

$\cos\theta = \frac{a}{c}$

$0 < \theta < \frac{\pi}{2}$



$x = \cos\theta$

$y = \sin\theta$

$\theta \in \mathbb{R}$

1.2. Alternatively, $y = \cos x$ is defined as the unique solution of

$$y'' + y = 0, \quad \text{with} \quad y(0) = 1, \quad y'(0) = 0$$

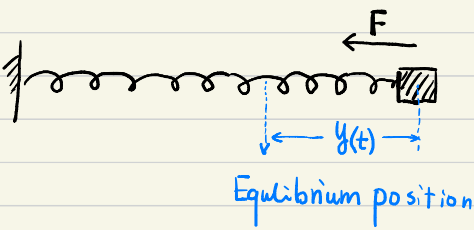
Similarly, $y = \sin x$ is the unique solution of

$$y'' + y = 0 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 1.$$

1.3 Simple harmonic motions

Here we see that sine/cosine functions can be used to describle simple harmonic motions.

Consider a spring linked with a mass M.



By Hookie's law, $F = -k\, y(t)$

(k is the spring constant)

By Newton's law
$$F = m\, y''(t)$$

Hence we have the following equation for the motion

$$y''(t) + \frac{k}{m} y(t) = 0.$$

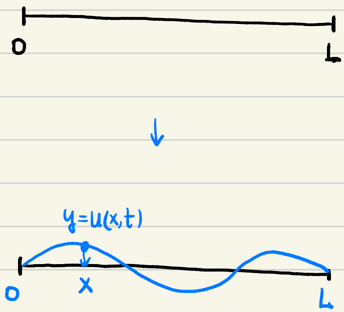
The general solution of the above equation is

$$y(t) = A \cdot \cos\left(\sqrt{\frac{k}{m}}\, t\right) + B \sin\left(\sqrt{\frac{k}{m}}\, t\right).$$

where A, B are constants.

1.4 The wave equation for a vibrating string

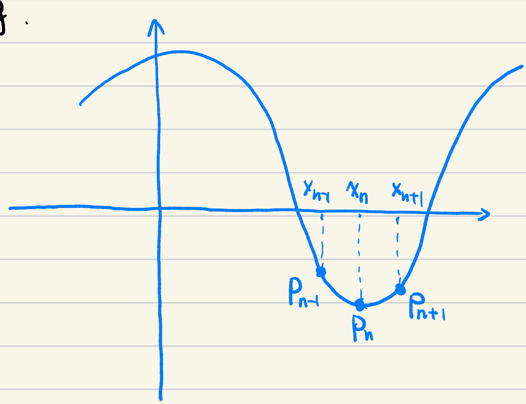
Consider a string fixed at its endpoints, vibrating freely and slightly



At time t.

Image that the string is made of many particles located at $x_0, x_1, x_2, \cdots$ with $x_{i+1} = x_i + h$

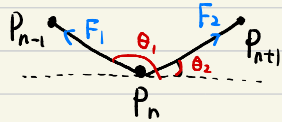
Each of which has mass ρh, where ρ is the density of the string.



set $y_n = u(x_n, t)$.

The force on P_n are given by the tensions between P_n and P_{n-1}, and between P_n, P_{n+1}.

The force acting on P_n (along the y-direction) is equal to



$$\tau \sin\theta_1 + \tau \sin\theta_2$$

$$\approx \tau \tan\theta_1 + \tau \tan\theta_2 \qquad \text{(since } \theta_1, \theta_2 \text{ are very small).}$$

$$= \tau \frac{y_{n-1} - y_n}{h} + \tau \cdot \frac{y_{n+1} - y_n}{h}$$

Then by Newton's law,

$$\rho h \cdot y_n''(t) = \frac{\tau}{h}\left(y_{n-1}(t) + y_{n+1}(t) - 2y_n(t) \right)$$

That is,

$$y_n''(t) = \frac{\tau}{\rho} \frac{u(x_n - h, t) + u(x_n - h, t) - 2u(x_n, t)}{h^2}$$

Assume that $u \in C^2([0, L] \times \mathbb{R}_+)$, where $\mathbb{R}_+ := (0, \infty)$.

Letting $h \to 0$, we have

$$\frac{\partial^2}{\partial t^2} u(x_n, t) = \frac{\tau}{\rho} \cdot \frac{\partial^2}{\partial x^2} u(x_n, t)$$

(Here we use the fact that

$$\frac{g(x+h)+g(x-h)-2g(x)}{h^2} \to g''(x) \quad \text{for } g\in C^2(\mathbb{R}) \text{ and } h\to 0.$$

Since x_n is an arbitrarily given point, we obtain

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\tau}{\rho}\,\frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0<x<L,\ t>0.$$

i.e.

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}, \qquad 0\le x\le L,\quad t>0, \qquad (*)$$

where $c=\sqrt{\frac{\tau}{\rho}}$. We call it the wave equation.

Normalization: set $X = \frac{\pi x}{L}$

$$T = \frac{c\pi t}{L}$$

and $U(X,T) = u(x,t)$.

Then

$$\frac{\partial^2 U}{\partial T^2} = \frac{\partial^2 U}{\partial X^2}, \qquad 0<X<\pi, \quad T>0.$$

1.5 The standing waves.

Def. A standing wave is a solution of the wave equation

$$(1) \quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0 \\ u(0,t) = u(\pi,t) = 0 \end{cases}$$

of the form

$$u = \varphi(x)\,\psi(t).$$

Prop. Any non-zero standing wave to Eq (1) must be of the form

$$u = (A \cos mt + B \sin mt) \sin mx, \quad m = 1, 2, \cdots$$

Pf. Plug $u = \varphi(x)\psi(t)$ into (1). We have

$$\varphi(x)\,\psi''(t) = \varphi''(x)\,\psi(t)$$

Hence

$$\frac{\varphi''(x)}{\varphi(x)} = \frac{\psi''(t)}{\psi(t)} = \lambda,$$

where λ is a constant.

Hence $$\varphi''(x) - \lambda\varphi(x) = 0.$$
$$\psi''(t) - \lambda\psi(t) = 0.$$

For the equation $\varphi''(x) - \lambda\varphi(x) = 0$, the general solution is given by

$$\varphi(x) = \begin{cases} a e^{\sqrt{\lambda} x} + b e^{-\sqrt{\lambda} x} & \text{if } \lambda > 0 \\ ax + b & \text{if } \lambda = 0 \\ a \cos(\sqrt{-\lambda}\, x) + b \sin(\sqrt{-\lambda}\, x) & \text{if } \lambda < 0. \end{cases}$$

Since $\varphi(0) = \varphi(\pi) = 0$, the existence of a non-zero solution

implies that $\lambda < 0$ and $\sin(\sqrt{-\lambda}\,\pi) = 0$,

which holds ~~if~~ if and only if $\lambda = -m^2$ for $m \in \mathbb{Z}_{>0}$.

i.e. $\varphi(x) = b \sin mx$ for some integer $m > 0$.

Correspondingly, $\psi''(t) + m^2 \psi(t) = 0$, which has a general solution

$$\psi(t) = (A \cos mt + B \sin mt).$$

It follows that

$$u = \varphi(x)\psi(t) = (a \cos mt + b \sin mt) \sin mx.$$

$\blacksquare$

1.6. Superposition of standing waves.

Now let us consider the wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, & 0 \le x \le \pi, \quad t > 0 \quad (1) \\ u(0,t) = u(\pi,t) = 0 & (2) \\ u(x,0) = f(x), \quad \dfrac{\partial}{\partial t} u(x,0) = 0 & (3) \end{cases}$$

(Initial condition)

It is known that a standing wave satisfies (1) and (2).

By linearity, any superposition of standing waves,

$$u(x,t) = \sum_{m=1}^{\infty} (A_m \cos mt + B_m \sin mt) \sin mx$$

also satisfies (1) and (2). To require that u satisfies (3), formally, we must require $B_m = 0$ and

$$\sum_{m=1}^{\infty} A_m \sin(mx) = f(x). \qquad (*)$$

Now it arises a natural question: for a given function f, Can we find coefficients A_m such that the above equality holds?

Suppose $(*)$ holds.

To guess a formula of A_m, we multiple both sides of $(*)$ by $\sin mx$ and integrate over $[0, \pi]$. Working formally, We have

$$\int_0^\pi f(x) \sin mx \, dx = \int_0^\pi \sum_{n=1}^{\infty} A_n \sin nx \sin mx \, dx$$

$$= \sum_{n=1}^{\infty} A_n \int_0^\pi \sin nx \sin mx \, dx$$

$$= A_m \cdot \frac{\pi}{2},$$

using $$\int_0^\pi \sin nx \sin mx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n. \end{cases}$$

That is, $$A_m = \frac{2}{\pi} \int_0^\pi f(x) \sin mx \, dx.$$

We call A_m the m-th Fourier Sine coefficient of f.